

# A Note on Doubly Warped Product Contact $CR$ -Submanifolds in trans-Sasakian Manifolds

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## Abstract

Warped product  $CR$ -submanifolds in Kählerian manifolds were intensively studied only since 2001 after the impulse given by B.Y. Chen in [2], [3]. Immediately after, another line of research, similar to that concerning Sasakian geometry as the odd dimensional version of Kählerian geometry, was developed, namely warped product contact  $CR$ -submanifolds in Sasakian manifolds (cf. [6], [7]). In this note we proved that there exists no proper doubly warped product contact  $CR$ -submanifolds in trans-Sasakian manifolds.

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## 1 Preliminaries

Let  $M$  be a Riemannian manifold isometrically immersed in a trans-Sasakian manifold  $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ .  $M$  is called a *contact  $CR$ -submanifold* if there exists on  $M$  a differentiable  $\phi$ -invariant distribution  $\mathcal{D}$  (i.e.  $\phi_x \mathcal{D}_x \subset \mathcal{D}_x$ , for all  $x \in M$ ) whose orthogonal complement  $\mathcal{D}^\perp$  of  $\mathcal{D}$  in  $T(M)$  is a  $\phi$ -anti-invariant distribution on  $M$ , i.e.  $\phi_x \mathcal{D}_x^\perp \subset T(M)_x^\perp$  for all  $x \in M$ . Here  $T(M)^\perp \longrightarrow M$  is the normal bundle of  $M$ .

It is customary to require that  $\xi$  be tangent to  $M$  than normal (in case when  $\widetilde{M}$  is Sasakian, by Prop. 1.1, K.Yano & M.Kon [10], p.43, in this later case,  $M$  must be  $\phi$ -anti-invariant, i.e.  $\phi_x T_x(M) \subset T(M)_x^\perp$ ,  $\forall x \in M$ ).

Given a contact  $CR$ -submanifold  $M$  of a trans-Sasakian manifold  $\widetilde{M}$ , either  $\xi \in \mathcal{D}$ , or  $\xi \in \mathcal{D}^\perp$  (the result holds only due the almost contact structure conditions). Therefore, the tangent space at each point decomposes orthogonally as

$$T(M) = H(M) \oplus \mathbf{R}\xi \oplus E(M) \quad (1)$$

where  $\phi H(M) = H(M)$ ,  $\phi^2|_{H(M)} = -I_{H(M)}$  ( $H(M)$  is called *the Levi distribution* of  $M$ ) and  $\phi E(M) \subset T(M)^\perp$ . We will consider only *proper* contact  $CR$ -submanifolds ( $\dim H(M) > 0$ ), i.e.  $M$  is neither  $\phi$ -invariant, nor  $\phi$ -anti invariant. Remark that both  $\mathcal{D} = H(M)$  and  $\mathcal{D} = H(M) \oplus \mathbf{R}\xi$  organize  $M$  as a contact  $CR$ -submanifold, but  $H(M)$  is never integrable if  $d\eta$  is non-degenerate (for example if the ambient is a contact manifold, e.g. a Sasakian manifold).

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An *almost contact structure*  $(\phi, \xi, \eta)$  on a  $(2m+1)$ -dimensional manifold  $\widetilde{M}$ , is defined by  $\phi \in \mathcal{T}_1^1(\widetilde{M})$ ,  $\xi \in \chi(\widetilde{M})$ ,  $\eta \in \Lambda^1(\widetilde{M})$  satisfying the following properties:  $\phi^2 = -I + \eta \otimes \xi$ ,  $\phi\xi = 0$ ,  $\eta \circ \phi = 0$ ,  $\eta(\xi) = 1$ . If on  $\widetilde{M}$  we have the metric  $\widetilde{g}$ , then the compatibility condition is required, namely,  $\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \eta(X)\eta(Y)$ .

An almost contact metric structure  $(\phi, \xi, \eta, \widetilde{g})$  on  $\widetilde{M}$  is called a *trans-Sasakian structure* if  $(\widetilde{M} \times \mathbf{R}, J, G)$  belongs to the class  $\mathcal{W}_4$  of the Gray-Hervella classification of almost Hermitian manifolds (see [4]), where  $J$  is the almost complex structure on the product manifold  $\widetilde{M} \times \mathbf{R}$  defined by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right)$$

for all vector fields  $X$  on  $\widetilde{M}$  and smooth functions  $f$  on  $\widetilde{M} \times \mathbf{R}$ . Here  $G$  is the product metric on  $\widetilde{M} \times \mathbf{R}$ . If  $\widetilde{\nabla}$  denotes the Levi Civita connection (of  $\widetilde{g}$ ), then the following condition

$$(\widetilde{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \quad (2)$$

holds for some smooth functions  $\alpha$  and  $\beta$  on  $\widetilde{M}$ . We say that the trans-Sasakian structure is of *type*  $(\alpha, \beta)$ . From the formula (2) it follows that

$$\widetilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \quad (3)$$

(See e.g. [1] for details.)

We note that trans-Sasakian structures of type  $(0, 0)$  are cosymplectic, trans-Sasakian structures of type  $(0, \beta)$  are  $\beta$ -Kenmotsu and a trans-Sasakian structure of type  $(\alpha, 0)$  are  $\alpha$ -Sasakian.

For any  $X \in \chi(M)$  we put  $PX = \tan(\phi X)$  and  $FX = \text{nor}(\phi X)$ , where  $\tan_x$  and  $\text{nor}_x$  are the natural projections associated to the direct sum decomposition  $T_x(\widetilde{M}) = T_x(M) \oplus T(M)_x^\perp$ ,  $x \in M$ . We recall the Gauss formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (4)$$

for any  $X, Y \in \chi(M)$ . Here  $\nabla$  is the induced connection and  $h$  is the second fundamental form of the given immersion. Since  $\xi$  is tangent to  $M$  we have

$$P\xi = 0, \quad F\xi = 0, \quad \nabla_X \xi = -\alpha PX + \beta(X - \eta(X)\xi), \quad h(X, \xi) = -\alpha FX, \quad X \in \chi(M). \quad (5)$$

## 2 Doubly Warped Products

Doubly warped products can be considered as generalization of warped products. A *doubly warped product*  $(M, g)$  is a product manifold of the form  $M =_f B \times_b F$  with the metric  $g = f^2 g_B \oplus b^2 g_F$ , where  $b : B \rightarrow (0, \infty)$  and  $f : F \rightarrow (0, \infty)$  are smooth maps and  $g_B, g_F$  are the metrics on the Riemannian manifolds  $B$  and  $F$  respectively. (See for example [9].) If either  $b \equiv 1$  or  $f \equiv 1$ , but not both, then we obtain a (*single*) *warped product*. If both  $b \equiv 1$  and  $f \equiv 1$ , then we have a *product manifold*. If neither  $b$  nor  $f$  is constant, then we have a *non trivial doubly warped product*.

If  $X \in \chi(B)$  and  $Z \in \chi(F)$ , then the Levi Civita connection is

$$\nabla_X Z = Z(\ln f)X + X(\ln b)Z. \quad (6)$$

We give the main result of this section. Namely we have

**Theorem 1** *There is no proper doubly warped product contact CR-submanifolds in trans-Sasakian manifolds.*

PROOF. Let  $M =_{f_2} N^\top \times_{f_1} N^\perp$  be a doubly warped product contact CR-submanifold in a trans-Sasakian manifold  $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ , i.e.  $N^\top$  is a  $\phi$ -invariant submanifold and  $N^\perp$  is a  $\phi$ -anti-invariant submanifold.

As we have already seen  $\xi \in \mathcal{D}$  or  $\xi \in \mathcal{D}^\perp$ .

Case 1.  $\xi \in \mathcal{D}$  i.e.  $\xi$  is tangent to  $N^\top$ . Taking  $Z \in \chi(N^\perp)$  we have  $\nabla_Z \xi = \beta Z$ . On the other hand, from (6), we get

$$\nabla_\xi Z = Z(\ln f_2)\xi + \xi(\ln f_1)Z.$$

It follows, since the two distributions are orthogonal, that

$$\begin{cases} \xi(\ln f_1) = \beta, \\ Z(\ln f_2) = 0 \quad \text{for all } Z \in \mathcal{D}^\perp. \end{cases} \quad (7)$$

The second condition yield  $f_2 \equiv \text{constant}$ , so, we cannot have doubly warped product contact CR-submanifolds of the form  $_{f_2} N^\top \times_{f_1} N^\perp$ , with  $\xi$  tangent to  $N^\top$ , other than warped product contact CR-submanifolds. Moreover, in this case  $\beta$  is a smooth function on  $N^\top$ .

Case 2.  $\xi \in \mathcal{D}^\perp$ , i.e.  $\xi$  is tangent to  $N^\perp$ . Taking  $X \in \chi(N^\top)$  we have  $\nabla_X \xi = -\alpha PX + \beta X$  in one hand and  $\nabla_X \xi = X(\ln f_1)\xi + \xi(\ln f_2)X$  in the other hand. Since the two distributions are orthogonal, we immediately get

$$\begin{cases} \xi(\ln f_2)X = -\alpha PX + \beta X, \\ X(\ln f_1) = 0 \quad \text{for all } X \in \mathcal{D}. \end{cases} \quad (8)$$

The second condition in (8) shows that  $M$  is a CR-warped product between a  $\phi$ -anti invariant manifold  $N^\perp$  tangent to the structure vector field  $\xi$  and an invariant manifold  $N^\top$ . If  $\dim \mathcal{D} = 0$ , then  $M$  is a  $\phi$ -anti invariant submanifold in  $\widetilde{M}$ . Otherwise, one can choose  $X \neq 0$ , and thus  $X$  and  $PX$  are linearly independent. Using first relation in (8) one gets  $\alpha = 0$  and  $\beta = \xi(\ln f_2)$ . This means that the ambient manifold is  $\beta$ -Kenmotsu with  $\beta \in C^\infty(N^\perp)$ .

This ends the proof. ■

In 1992, J.C. Marero in ([5]) showed that a trans-Sasakian manifold of dimension  $\geq 5$  is either  $\alpha$ -Sasakian,  $\beta$ -Kenmotsu or cosymplectic. So, we can state

**Corollary 2** Let  $\widetilde{M}$  be

1. *either an  $\alpha$ -Sasakian manifold,*
2. *or a  $\beta$ -Kenmotsu manifold,*
3. *or a cosymplectic manifold.*

Then, there is no proper doubly warped product contact CR-submanifolds in  $\widetilde{M}$ . More precisely we have,

✓ if  $\xi \in \mathcal{D}$ :

$M = \widetilde{N}^\top \times_f N^\perp$ ,  $\xi$  is tangent to  $N^\top$  and  $f \in C^\infty(N^\top)$ , where  $\widetilde{N}^\top$  is the manifold  $N^\top$  with a homothetic metric  $c^2 g_{N^\top}$ , ( $c \in \mathbf{R}$ ). Moreover, in case 2,  $\beta$  is a smooth function on  $N^\top$ .

✓ if  $\xi \in \mathcal{D}^\perp$ :

1.  $M$  is a  $\phi$ -anti-invariant submanifold in  $\widetilde{M}$  ( $\dim \mathcal{D} = 0$ );

2-3.  $M = \widetilde{N}^\perp \times_f N^\top$ ,  $\xi$  is tangent to  $N^\perp$  and  $f \in C^\infty(N^\perp)$ , where  $\widetilde{N}^\perp$  is the manifold  $N^\perp$  with a homothetic metric  $c^2 g_{N^\perp}$ , ( $c \in \mathbf{R}$ ). Moreover, in case 2,  $\beta$  is a smooth function on  $N^\perp$

PROOF. The statements follow from the Theorem 1. ■

It follows that there is no warped product contact  $CR$  submanifolds in  $\alpha$ -Sasakian and cosymplectic manifolds in the form  $N^\perp \times_f N^\top$  (with  $\xi$  tangent to  $N^\top$ ) other than product manifolds ( $f$  must be a real positive constant) – after a homothetic transformation of the metric on  $N^\top$ . If the ambient is  $\beta$ -Kenmotsu, then

- there is no contact  $CR$ -product submanifolds
- there is no contact  $CR$ -warped product submanifolds of type  $N^\perp \times_f N^\top$  (with  $\xi$  tangent to  $N^\top$ ) (see also [8]).

We will give an example of warped product contact  $CR$ -submanifold of type  $N^\perp \times_f N^\top$  in Kenmotsu manifold, with  $\xi$  tangent to  $N^\perp$ :

Consider the complex space  $\mathbf{C}^m$  with the usual Kaehler structure and real global coordinates  $(x^1, y^1, \dots, x^m, y^m)$ . Let  $\widetilde{M} = \mathbf{R} \times_f \mathbf{C}^m$  be the warped product between the real line  $\mathbf{R}$  and  $\mathbf{C}^m$ , where the warping function is  $f = e^z$ ,  $z$  being the global coordinate on  $\mathbf{R}$ . Then,  $\widetilde{M}$  is a Kenmotsu manifold (see e.g. [1]). Consider the distribution  $\mathcal{D} = \text{span} \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s} \right\}$  and  $\mathcal{D}^\perp = \text{span} \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial x^{s+1}}, \dots, \frac{\partial}{\partial x^m} \right\}$  which are integrable and denote by  $N^\top$  and  $N^\perp$  the integral submanifolds, respectively. Let  $g_{N^\top} = \sum_{i=1}^s ((dx^i)^2 + (dy^i)^2)$  and  $g_{N^\perp} = dz^2 + e^{2z} \sum_{a=s+1}^m (dx^a)^2$  be Riemannian metrics on  $N^\top$  and  $N^\perp$ , respectively. Then,  $M = N^\perp \times_f N^\top$  is a contact  $CR$ -submanifold, isometrically immersed in  $\widetilde{M}$ . Here, the warping function is  $f = e^z$ .

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